A Systematic Approach to the Derivation of Constitutive Parameters of a Perfectly Matched Absorber

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***Abstract*—In this letter, the constitutive parameters of a perfectly matched absorber are derived by imposing the condition that the fields decay within the absorber in a required manner, and it is shown that this absorber can be realized as a bianisotropic medium. This strategy is similar to that employed for the realization of perfectly matched layers (PML's), via the use of anisotropic media, and the imposition of certain conditions on the complex permittivity and permeability tensors. However, unlike the PML, the bianisotropic media are desirable candidates for *physically realizable* and *passive absorbers*. Furthermore, theoretically, such media have the potential of absorbing incident waves of arbitrary frequency and direction of propagation with no reflection.**

I. Introduction

PERFECTLY matched layers (PML's) have been widely used for mesh truncation in finite methods, both in time- and frequency-domain applications [1]–[3]. The PML region absorbs an incident plane wave with no reflection irrespective of its angle of incidence, and the transmitted wave is attenuated, within the layer, in the direction normal to the interface. Anisotropic perfectly matched absorbers have been introduced [2] as alternatives to the split-field PML formulation of Berenger [4], where the fields within the region are non-Maxwellian (which implies that this medium cannot be realized physically). The permittivity and permeability tensors of such matched absorbers satisfy

$$\bar{\bar{\varepsilon}} = \varepsilon_0[\Lambda] \qquad \bar{\bar{\mu}} = \mu_0[\Lambda] \tag{1}$$

where $[\Lambda]$ is a diagonal matrix given by

$$[\Lambda] = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \tag{2}$$

where $a = 1 - j(\sigma/\omega\varepsilon_0)$, and it is assumed that the PML/free space interface is the plane $x = 0$. This formulation provides a basis for the physical realization of a perfectly matched absorber as a uniaxial magneto-dielectric material. One of the elements of (2), however, has a negative conductivity, and, hence, this material is active. This, in turn, precludes its physical realization with passive inclusions.

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In a recent publication, Tretyakov [5] has suggested that material absorbers can be synthesized with uniaxial Omega composites. These media are bianisotropic, and are realized by embedding Ω-shaped metallic inclusions in a background dielectric medium. The author of [5] has also pointed out that it may be possible to realize a PML with fewer restrictions on the permittivity and permeability tensors.

Another approach to mesh truncation in finite methods is the transparent absorbing boundary (TAB) method [6], where the open-space Maxwell's equations are transformed into an equivalent system with a closed homogeneous boundary. This is achieved by modulating the amplitudes of the fields in the computational domain to obtain reflection-free truncation of this domain. In this letter, we demonstrate that the numerical artifice embodied in TAB can be physically realized as a medium with suitably defined constitutive parameters. We show that the absorbing layer in TAB is a bianisotropic medium, with spatially varying constitutive parameters that are governed by the scalar function used to truncate the fields.

II. Derivation of the Constitutive Parameters of the Perfectly Matched Absorbing Medium

In a linear medium, the constitutive relations are given by [7]

$$\vec{D} = \bar{\bar{\varepsilon}}.\vec{E} + \bar{\bar{\xi}}.\vec{H} \tag{3}$$

$$\vec{B} = \bar{\bar{\zeta}}.\vec{E} + \bar{\bar{\mu}}.\vec{H} \tag{4}$$

where the dyadic parameters $\bar{\bar{\varepsilon}}$, $\bar{\bar{\xi}}$, $\bar{\bar{\zeta}}$ and $\bar{\bar{\mu}}$ depend on the properties of the medium. Such a medium is known as a bianisotropic medium since $\vec{D}$ and $\vec{B}$ are related to both $\vec{E}$ and $\vec{H}$. If the dyadics are functions of space variables, the medium is inhomogeneous. If they depend on frequency, the medium is said to be time-dispersive, and if they contain space derivatives, the medium is space-dispersive, or nonlocal. These definitions are important in the characterization of the medium as an absorber.

Conventionally, treatises on matched absorbers begin with either differential equations or postulates of the constitutive relations of an anisotropic media and demonstrate that the fields decay in such media in a desirable manner to provide the absorption characteristics. In this work we begin, instead, by postulating the nature of the field variation within the absorbing medium and then derive its constitutive relations in a systematic manner. Toward this end, we start by choosing a scalar modulation function that would yield the requisite decay behavior of the field in the absorbing medium. Next, we derive the second-order partial differential equations whose solutions exhibit the desired decay behavior, and we follow this up by rewriting these partial differential equations in the format of Maxwell's equations. As a final step, we deduce the constitutive relations from the above Maxwell's equations.

We start by assuming that the half-space $x < 0$ is free space, and that the field in this domain is a plane wave given by

$$E_z(x, y) = e^{-jk[\cos\theta x + \sin\theta y]} \tag{5}$$

where k is the wave number and θ is the angle of incidence measured from the x axis. We stipulate that, in the region $x > 0$, $E_z(x, y)$ be given by

$$E_z(x, y) = f(x) e^{-jk[\cos\theta x + \sin\theta y]} \tag{6}$$

where the function $f(x)$ satisfies the following conditions:

a) $f(0) = 1$;
b) $f(x)$ is a monotonically decreasing function of x (i.e., $f'(x) \leq 0$).

It is evident that condition a) assures the continuity of E_z at the interface $x = 0$, and b) is a statement of the fact that E_z should decay in the region $x > 0$. In [6], the function $f(.)$ is chosen to be

$$f(x) = \left[1 - \left(\frac{x}{L_x}\right)^m\right]^n \tag{7}$$

where L_x is the length of the attenuation path and m and n are integers such that $m, n \geq 1$. For this choice, it is evident that the TAB medium is terminated by a perfectly conducting wall at $x = L_x$ (i.e., $E_z(L_x, y) = 0$). Without any loss of generality, however, we can assume that the field satisfies (6) in the half-space $x > 0$, and this allows the function $f(.)$ to be defined over the entire positive half-space. For instance, we can define $f(x) = e^{-\beta x}$, where β is a positive constant.

The next step in our derivation is to determine the partial differential equations whose solutions yield the desired $E_z(x, y)$ in both half-spaces. In the region $x < 0$, the plane wave expression (5) obviously satisfies the Helmholtz equation

$$\nabla^2 E_z + k^2 E_z = 0. \tag{8}$$

The partial differential equation satisfied by E_z in the region $x > 0$ can be obtained by dividing both sides of (6) by $f(x)$ (assuming that $f(x) \neq 0$ for $x > 0$), and operating with $(\nabla^2 + k^2)$ on both terms. Noting that the right-hand side vanishes, we obtain

$$\frac{\partial}{\partial x}\left[\frac{\partial E_z}{\partial x} - \frac{f'(x)}{f(x)} E_z\right] + \frac{\partial^2 E_z}{\partial y^2} + k^2 E_z - \frac{f'(x)}{f(x)} \frac{\partial E_z}{\partial x} + \left[\frac{f'(x)}{f(x)}\right]^2 E_z = 0. \tag{9}$$

From (9), Maxwell's curl equations can now be extracted. We derive, for instance, the following curl-type equations:

$$-\frac{\partial E_z}{\partial y}\hat{a}_x + \left(\frac{\partial E_z}{\partial x} - \frac{f'(x)}{f(x)} E_z\right)\hat{a}_y = j\omega\mu_0 [H_x \hat{a}_x + H_y \hat{a}_y] \tag{10}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \frac{f'(x)}{f(x)} H_y = j\omega\varepsilon_0 E_z. \tag{11}$$

Next, we express (10) and (11) in a more compact form as

$$-\nabla \times \vec{E} + \frac{f'(x)}{f(x)}\hat{a}_x \times \vec{E} = j\omega\mu_0 \vec{H} \tag{12}$$

$$\nabla \times \vec{H} - \frac{f'(x)}{f(x)}\hat{a}_x \times \vec{H} = j\omega\varepsilon_0 \vec{E}. \tag{13}$$

Finally, to obtain the constitutive relations, we further reduce (12) and (13) to get

$$-\nabla \times \vec{E} = j\omega\left[\mu_0 \vec{H} + \frac{j}{\omega}\frac{f'(x)}{f(x)}\hat{a}_x \times \vec{E}\right] \tag{14}$$

$$\nabla \times \vec{H} = j\omega\left[\varepsilon_0 \vec{E} - \frac{j}{\omega}\frac{f'(x)}{f(x)}\hat{a}_x \times \vec{H}\right]. \tag{15}$$

Noting that $k = \omega\sqrt{\varepsilon_0\mu_0}$, we can now express the constitutive relations in this medium as

$$\vec{D} = \varepsilon_0 \vec{E} - j\sqrt{\varepsilon_0\mu_0}\,\frac{1}{k}\,\frac{f'(x)}{f(x)}\hat{a}_x \times \vec{H} \tag{16}$$

$$\vec{B} = \mu_0 \vec{H} + j\sqrt{\varepsilon_0\mu_0}\,\frac{1}{k}\,\frac{f'(x)}{f(x)}\hat{a}_x \times \vec{E}. \tag{17}$$

A comparison of (16) and (17) with the constitutive relations of a bianisotropic medium, given by (3) and (4), leads us to the following results for the constitutive parameters:

$$\overline{\overline{\varepsilon}} = \varepsilon_0 \overline{\overline{I}} \tag{18}$$

$$\overline{\overline{\mu}} = \mu_0 \overline{\overline{I}} \tag{19}$$

where $\overline{\overline{I}}$ is the identity matrix. All entries of $\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}$ are zero, except for ξ_{zy} and ζ_{yz}, which are given by

$$\xi_{zy} = \zeta_{yz} = -j\sqrt{\varepsilon_0\mu_0}\,\frac{1}{k}\,\frac{f'(x)}{f(x)}. \tag{20}$$

Since, by definition, $f(x)$ is a decreasing function of x, $f'(x) \leq 0$; hence $\mathrm{Im}(\xi_{zy}) = \mathrm{Im}(\zeta_{yz}) \geq 0$, which, in turn, accounts for the loss mechanism in TAB.

Another important observation related to (20) is the appearance of k in the denominator, leading us to the conclusion that the medium is time dispersive. This term implies that larger values of the parameters ξ_{zy} and ζ_{yz} must be chosen at lower frequencies to achieve effective attenuation in the field quantities. For instance, if we let $f(x) = e^{-\beta x}$, (20) becomes

$$\xi_{zy} = \zeta_{yz} = j\sqrt{\varepsilon_0\mu_0}\,\frac{\beta}{k}. \tag{21}$$

This is an important result, because it is well known that the attenuation within the medium in an anisotropic PML depends on frequency and the angle of incidence [2]. In TAB, $f(x)$ can be chosen to be independent of these two parameters. However, in a nondispersive bianisotropic absorber, with constant $\xi_{zy} = \zeta_{yz} = j\sqrt{\varepsilon_0\mu_0}\,\beta$, $f(x)$ becomes

$$f(x) = e^{-k\beta x} \tag{22}$$

which is obviously, frequency dependent.

III. Conclusions

In this letter, we have derived the constitutive relations of an absorbing medium by postulating certain decay behavior of the fields within the medium. Important contributions of our letter are the derivation of the partial differential equations whose solutions yield the required decaying field, and the extraction of the constitutive relations from the Maxwell's equations derived from the above partial differential equations. We have demonstrated that the constitutive relationships, thus obtained, are identical to those of a bianisotropic medium. An important consequence of this is that it offers us the potential for realizing bianisotropic perfectly matched absorbers as generalizations of anisotropic PML absorbers.

By showing that our equations are identical to those employed in the TAB method, we have established the equivalence between the above approach and the bianisotropic media concept. This enables us to control the decay behavior of the transmitted field via the choice of the scalar truncation function $f(x)$, which also determines the $\overline{\overline{\xi}}$ (or $\overline{\overline{\zeta}}$) parameter of the medium. In TAB, the permittivity and permeability parameters are chosen to be ε_0 and μ_0, but it should be possible to generalize this and derive a perfectly matched absorber by using a material with tensor parameters $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$, in conjunction with the parameters $\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}$. The import of this result on the physical realizability of perfectly matched bianisotropic absorbers is currently under investigation by the authors.

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